The elastodynamic Liénard-Wiechert potentials and elastic fields of non-uniformly moving point and line forces

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Abstract

The purpose of this paper is to investigate the fundamental problem of the non-uniform subsonic motion of a point force and line forces in an unbounded, homogeneous, isotropic medium in analogy to the electromagnetic Liénard-Wiechert potentials. The exact closed-form solutions of the displacement and elastic fields produced by the point force and line forces are calculated. The displacement fields can be identified with the elastodynamic Liénard-Wiechert tensor potentials. For a non-uniformly moving point force, we decompose the elastic fields into a radiation part and a non-radiation part. We show that the solution of a non-uniformly moving point force is the generalization of the Stokes solution towards the non-uniform motion. For line forces the mathematical solutions are given in the form of time-integrals and, therefore, their motion depends on the history.

Keywords: non-uniform motion; point force; line forces; elastodynamics; radiation; retardation; elastic waves.

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1 Introduction

An important item in elastodynamics is concerned with the radiation and the waves produced by the non-uniform motion of body forces. This is a fascinating and interdisciplinary research topic. The radiation problem has attracted the interest of researchers from different fields such as applied mathematics, material science, continuum mechanics, and seismology (see, e.g, [1, 19, 12, 2, 20]). A fundamental question is: what is the elastic radiation caused by non-uniformly moving point forces?

In elastostatics, the so-called Kelvin problem concerns with the displacement and elastic fields produced by a static point force. In elastodynamics the displacement field generated by a time-dependent concentrated point load was first presented by Stokes [22] (see, e.g., [1, 2, 11]). In the Stokes problem, the body force is considered as a concentrated load of time-dependent magnitude. The Stokes solution can be considered as the first mathematical model of an earthquake [4]. Concentrated line forces with time-dependent magnitude were studied by de Hoop [6] and Achenbach [1]. The wave-motion caused by a line force moving non-uniformly in a fixed direction was considered by Freund [10]. A non-uniformly moving line force in an anisotropic elastic solid was studied by Wu [23].

The radiation problem of point forces is three-dimensional so that Huygens' principle prevails. Using the Helmholtz decomposition, the so-called retarded potentials were given for the waves produced by body forces in elastodynamics (see, e.g., [1, 19]). A more general expression for the retarded potential in elastodynamics was given by Hudson [12]. Elastodynamic fields propagate with finite velocities. There always is a time-delay before a change in elastodynamic conditions initiated at a point of space can produce an effect at any other point of space. This time-delay is called elastodynamic retardation.

In electrodynamics, radiation is caused by the non-uniform motion of an electric point charge. The electric and magnetic potentials of such a non-uniformly moving point charge are called the Liénard-Wiechert potentials. The corresponding electric and magnetic field strengths consist of velocity-depending fields and acceleration-depending fields. The last ones are the fields of radiation. This is a standard topic in electromagnetic field theory and is covered in a lot of books on electrodynamics (e.g. [15, 13]). It is quite surprising that nothing has been investigated in this direction in the elastodynamics of moving point forces. No solution of a non-uniformly moving point force analogous to the Liénard-Wiechert potential can be found in standard books on elastic waves (e.g. [1, 19, 9, 7, 12, 2, 20]).

The purpose of the present paper is to investigate the fundamental problem of the non-uniform motion of a point force as well as line forces in an unbounded, homogeneous, isotropic medium in analogy to the electromagnetic Liénard-Wiechert potentials. We consider the subsonic motion. The paper is organized as follows. In Section 2, we present the framework of elastodynamics and we formulate the equation of motion. In Section 3, using the three-dimensional elastodynamic Green tensor, we calculate the elastodynamic Liénard-Wiechert potential of a point force. In Section 4, using the Liénard-Wiechert potential of a point force, we determine the elastic distortion and the velocity fields (particle velocity) of the medium caused by the non-uniformly moving point force. In addition, we specify the radiation fields proportional to the acceleration of the point force. The limit to the Stokes solution is performed in Section 5. The static limit of the displacement and elastic fields of the non-uniformly moving point force is given in Section 6. In Section 7, using the two-dimensional Green tensors, we give the general solution of the two-dimensional non-uniformly moving line forces. We close the paper with conclusions in Section 8.

2 The elastodynamic equation of motion

In elastodynamics [11], the force balance law reads[1]

$$\dot{p}_i - \partial_j \sigma_{ij} = F_i\,, \tag{1}$$

where $\boldsymbol{p}$, $\boldsymbol{\sigma}$ and $\boldsymbol{F}$ are the linear momentum vector, the force stress tensor and the body force vector. In the theory of linear elasticity, the momentum vector $\boldsymbol{p}$ and the stress tensor $\boldsymbol{\sigma}$ can be expressed in terms of the physical state quantities, namely, the velocity vector (particle velocity) $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and the elastic distortion tensor $\boldsymbol{\beta} = (\mathrm{grad}\,\boldsymbol{u})^{\mathrm{T}}$ of the medium which can be derived from a displacement vector $\boldsymbol{u}$ by means of the following constitutive relations

$$p_i = \rho\, v_i = \rho\, \dot{u}_i\,, \tag{2}$$

$$\sigma_{ij} = C_{ijkl}\,\beta_{kl} = C_{ijkl}\,\partial_l u_k\,, \tag{3}$$

where ρ denotes the mass density and C_{ijkl} is the tensor of elastic moduli. The tensor C_{ijkl} possesses the following symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}\,. \tag{4}$$

If we substitute the constitutive relations (2) and (3) in Eq. (1), we obtain the force balance law expressed in terms of the displacement vector $\boldsymbol{u}$

$$\rho\, \ddot{u}_i - C_{ijkl}\partial_j\partial_l u_k = F_i\,. \tag{5}$$

The solution of Eq. (5) can be represented as a convolution-integral in space and time. In an unbounded medium and under the assumption of zero initial conditions, which means that $\boldsymbol{u}(\boldsymbol{r}, t_0)$ and $\dot{\boldsymbol{u}}(\boldsymbol{r}, t_0)$ are zero for $t_0 \to -\infty$, the solution of $\boldsymbol{u}$ reads

$$u_i(\boldsymbol{r}, t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} G_{ij}(\boldsymbol{r}-\boldsymbol{r}', t-t')\, F_j(\boldsymbol{r}', t')\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}t'\,. \tag{6}$$

Here, G_{ij} is the elastodynamic Green tensor of the anisotropic Navier equation defined by

$$\big[\delta_{ik}\,\rho\,\partial_{tt} - C_{ijkl}\partial_j\partial_l\big] G_{km} = \delta_{im}\,\delta(t)\delta(\boldsymbol{r})\,, \tag{7}$$

where $\delta(.)$ denotes the Dirac delta function and δ_{ij} is the Kronecker delta. The tensor of elastic moduli for isotropic materials is given by

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big)\,, \tag{8}$$

where λ and μ are the Lamé constants. Substituting Eq. (8) in Eqs. (5) and (7), we obtain respectively the isotropic Navier equations for the displacement vector

$$\big[\delta_{ij}\,\rho\,\partial_{tt} - \delta_{ij}\,\mu\,\Delta - (\lambda+\mu)\,\partial_i\partial_j\big] u_j = F_i\,, \tag{9}$$

and for the elastodynamic Green tensor

$$\big[\delta_{ij}\,\rho\,\partial_{tt} - \delta_{ij}\,\mu\,\Delta - (\lambda+\mu)\,\partial_i\partial_j\big] G_{jm} = \delta_{im}\,\delta(t)\delta(\boldsymbol{r})\,, \tag{10}$$

[1] Spatial differentiation is denoted by $\partial_j \equiv \partial/\partial x_j$, and for the differentiation with respect to time t we use the notation $\dot{p}_i \equiv \partial_t p_i$.

where Δ denotes the Laplacian.

When the material is isotropic and infinitely extended, the three-dimensional elastodynamic Green tensor reads [18, 9, 2, 20]

$$G_{ij}(\boldsymbol{r},t) = \frac{1}{4\pi\rho}\left\{\frac{\delta_{ij}}{rc_{\mathrm{T}}^2}\,\delta(t-r/c_{\mathrm{T}}) + \frac{x_i x_j}{r^3}\left(\frac{1}{c_{\mathrm{L}}^2}\,\delta(t-r/c_{\mathrm{L}}) - \frac{1}{c_{\mathrm{T}}^2}\,\delta(t-r/c_{\mathrm{T}})\right)\right.$$
$$\left. + \left(\frac{3x_i x_j}{r^2} - \delta_{ij}\right)\frac{1}{r^3}\int_{r/c_{\mathrm{L}}}^{r/c_{\mathrm{T}}} \tau\,\delta(t-\tau)\,\mathrm{d}\tau\right\}, \tag{11}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It should be pointed out that the tensor in Eq. (11) is the retarded Green tensor. Here c_{L} and c_{T} denote the velocities of the longitudinal and transversal elastic waves (sometimes called P- and S-waves). The two sound-velocities can be given in terms of the Lamé constants ($c_{\mathrm{T}} < c_{\mathrm{L}}$)

$$c_{\mathrm{L}} = \sqrt{\frac{2\mu+\lambda}{\rho}}\,, \qquad c_{\mathrm{T}} = \sqrt{\frac{\mu}{\rho}}\,. \tag{12}$$

The elastodynamic Green tensor (11) is a tensor with support along the two sound-cones $r = c_{\mathrm{T}}t$ and $r = c_{\mathrm{L}}t$ as well as in between them. Moreover, it should be mentioned that it consists of near-field and far-field terms. The first two terms in Eq. (11) decay as $1/r$ and, thus, they are the far-field terms. The last term in Eq. (11) decays more rapidly as $1/r^2$ which gives the near-field term (see, e.g., [5]).

3 The elastodynamic Liénard-Wiechert potential of a point force

Now we consider the non-uniform motion of a point force of total strength $Q_j(t)$, situated at the position $\boldsymbol{s}(t)$. Then the point force is

$$F_i(\boldsymbol{r},t) = Q_i(t)\,\delta(\boldsymbol{r}-\boldsymbol{s}(t)) \qquad \text{for } \boldsymbol{r}\in\mathbb{R}^3,\ t\in\mathbb{R}\,. \tag{13}$$

Moreover, only subsonic source-speeds will be admitted ($|\boldsymbol{V}| < c_{\mathrm{T}}$). Substitution of Eq. (13) in Eq. (6) and integration in $\boldsymbol{r}'$ lead to

$$u_i(\boldsymbol{r},t) = \int_{-\infty}^{t} G_{ij}(\boldsymbol{r}-\boldsymbol{s}(t'),t-t')\,Q_j(t')\,\mathrm{d}t'\,. \tag{14}$$

The structure of the Green tensor (11) produces in Eq. (14) three characteristic integrals which we have to calculate. If we substitute the elastodynamical Green tensor (11) in Eq. (14) and use the relation

$$\frac{1}{r^2}\int_{r/c_{\mathrm{L}}}^{r/c_{\mathrm{T}}} \tau\,\delta(t-\tau)\,\mathrm{d}\tau = \int_{1/c_{\mathrm{L}}}^{1/c_{\mathrm{T}}} \kappa\,\delta(t-\kappa r)\,\mathrm{d}\kappa\,, \tag{15}$$

where κ is a dummy variable with the dimension of slowness = 1/[velocity], we obtain

$$u_i(\boldsymbol{r},t) = \frac{1}{4\pi\rho}\int_{-\infty}^{t}\left\{\frac{\delta_{ij}\,\delta(t-t'-R(t')/c_{\mathrm{T}})}{c_{\mathrm{T}}^2\,R(t')}\right.$$
$$+ \frac{R_i(t')R_j(t')}{R^3(t')}\left(\frac{1}{c_{\mathrm{L}}^2}\,\delta\big(t-t'-R(t')/c_{\mathrm{L}}\big) - \frac{1}{c_{\mathrm{T}}^2}\,\delta\big(t-t'-R(t')/c_{\mathrm{T}}\big)\right)$$
$$\left. + \left(\frac{3R_i(t')R_j(t')}{R^3(t')} - \frac{\delta_{ij}}{R(t')}\right)\int_{1/c_{\mathrm{L}}}^{1/c_{\mathrm{T}}} \kappa\,\delta(t-t'-\kappa\,R(t'))\,\mathrm{d}\kappa\right\} Q_j(t')\,\mathrm{d}t'\,, \tag{16}$$

where $\boldsymbol{R}(t') = \boldsymbol{r} - \boldsymbol{s}(t')$ and $R(t') = [R_m(t')R_m(t')]^{1/2} = |\boldsymbol{r} - \boldsymbol{s}(t')|$. Now the time-integration in Eq. (16) can be performed. We express the integrals in terms of retarded variables by appeal to the relation [21, 3]

$$\int \delta(f(t'))\, g(t')\, \mathrm{d}t' = \frac{g(t')}{|\mathrm{d}f/\mathrm{d}t'|}\bigg|_{\text{at } f(t')=0} . \tag{17}$$

Mathematically, the factor $1/|\mathrm{d}f/\mathrm{d}t'|$ is the Jacobian of the transformation from t' to the new integration variable $f(t')$. This mapping between the two variables is one-to-one if the Jacobian is different from zero. A sufficient condition for this is that the velocity of the source (point force) is less than the slowest wave speed ($|\boldsymbol{V}| < c_{\mathrm{T}}$). The first integral of Eq. (16) can be carried out with

$$\int \frac{\delta(t - t' - R(t')/c_{\mathrm{T}})\, Q_j(t')}{R(t')}\, \mathrm{d}t' = \frac{Q_j(t')}{R(t') - V_m(t')R_m(t')/c_{\mathrm{T}}}\bigg|_{t'=t_{\mathrm{T}}} , \tag{18}$$

where we introduced the so-called transversal retarded time $t_{\mathrm{T}} = t'(\boldsymbol{r}, t)$ which is the solution of the condition

$$t - t' - R(t')/c_{\mathrm{T}} = 0 . \tag{19}$$

$\boldsymbol{V} = \dot{\boldsymbol{s}}$ is the velocity of the moving body force and $\boldsymbol{R}$ is the distance vector from the position of the point force $\boldsymbol{s}$, the sender of elastic waves, to the point of the observer $\boldsymbol{r}$, the receiver of the elastic waves. The second integral is

$$\int \frac{R_i(t')R_j(t')\, Q_j(t')}{R^3(t')}\, \delta(t - t' - R(t')/c_{\mathrm{L,T}})\, \mathrm{d}t' = \frac{R_i(t')R_j(t')}{R^2(t')}\, \frac{Q_j(t')}{R(t') - V_m(t')R_m(t')/c_{\mathrm{L,T}}}\bigg|_{t'=t_{\mathrm{L,T}}} . \tag{20}$$

Here $t_{\mathrm{L}} = t'(\boldsymbol{r}, t)$ denotes the longitudinal retarded time which is the solution of the equation

$$t - t' - R(t')/c_{\mathrm{L}} = 0 . \tag{21}$$

We perform the third integral as follows

$$\begin{aligned} &\int \left(\frac{3R_i(t')R_j(t')}{R^3(t')} - \frac{\delta_{ij}}{R(t')}\right) Q_j(t') \int_{1/c_{\mathrm{L}}}^{1/c_{\mathrm{T}}} \kappa\, \delta(t - t' - \kappa\, R(t'))\, \mathrm{d}\kappa\, \mathrm{d}t' \\ &\qquad = \int_{1/c_{\mathrm{L}}}^{1/c_{\mathrm{T}}} \left(\frac{3R_i(t')R_j(t')}{R^2(t')} - \delta_{ij}\right) \frac{Q_j(t')\, \kappa\, \mathrm{d}\kappa}{R(t') - \kappa\, V_m(t')R_m(t')}\bigg|_{t'=t_\kappa} \end{aligned} \tag{22}$$

with $t_\kappa = t'(\boldsymbol{r}, t)$ as solution of the equation

$$t - t' - \kappa\, R(t') = 0 . \tag{23}$$

The retarded time t_κ is an effective retarded time for the κ-integration with limits $(1/c_{\mathrm{L}}, 1/c_{\mathrm{T}})$. The solutions of t_{T}, t_{L}, t_κ are unique if $|\boldsymbol{V}|$ is less than c_{T}. Thus, for subsonic motion the solutions of Eqs. (19), (21) and (23) are unique. The retarded times are a result of the finite speeds of propagation for elastodynamic waves. In Eqs. (18), (20) and (22) we have used the relation

$$\left|\frac{\mathrm{d}f(t')}{\mathrm{d}t'}\right|_{t'=t_{\mathrm{ret}}} = 1 - \frac{V_m(t')R_m(t')}{c\, R(t')}\bigg|_{t'=t_{\mathrm{ret}}} > 0 \quad \text{for } |\boldsymbol{V}| < c_{\mathrm{T}} , \quad c = c_{\mathrm{T}}, c_{\mathrm{L}}, 1/\kappa \quad \text{and } t_{\mathrm{ret}} = t_{\mathrm{T}}, t_{\mathrm{L}}, t_\kappa , \tag{24}$$

where $f(t') = t - t' - R(t')/c$.

Thus, carrying out the integration in t' in Eq. (16), we find the explicit expression for the displacement field of a non-uniformly moving point force which we call the elastodynamic Liénard-Wiechert potential of a point force

$$u_i(\boldsymbol{r},t) = \frac{1}{4\pi\rho}\left\{ \frac{1}{c_{\rm T}^2}\left[\left(\delta_{ij} - \frac{R_i(t')R_j(t')}{R^2(t')}\right)\frac{Q_j(t')}{R(t') - V_m(t')R_m(t')/c_{\rm T}}\right]\Big|_{t'=t_{\rm T}} + \frac{1}{c_{\rm L}^2}\left[\frac{R_i(t')R_j(t')}{R^2(t')}\,\frac{Q_j(t')}{R(t') - V_m(t')R_m(t')/c_{\rm L}}\right]\Big|_{t'=t_{\rm L}} + \int_{1/c_{\rm L}}^{1/c_{\rm T}} {\rm d}\kappa\,\kappa\left[\left(\frac{3R_i(t')R_j(t')}{R^2(t')} - \delta_{ij}\right)\frac{Q_j(t')}{R(t') - \kappa\, V_m(t')R_m(t')}\right]\Big|_{t'=t_\kappa}\right\}, \qquad (25)$$

where $\boldsymbol{R}(t')$, $\boldsymbol{V}(t')$ and $\boldsymbol{Q}(t')$ are to be evaluated at the corresponding retarded times. The first δ_{ij}-term in Eq. (25) has the form of the acoustic Liénard-Wiechert potential [3, 8] and if $\boldsymbol{Q}$ is time-independent, this term reduces to the form of the well-known electric Liénard-Wiechert potential of a point charge in electrodynamics (see, e.g., [15, 13]). Due to the appearance of two velocities of the elastic waves, the elastodynamic Liénard-Wiechert potential of a point force has a more complicated but rather straightforward structure. Eq. (25) consists of three characteristic pieces. The first term is the transversal one, transmitting with speed $c_{\rm T}$, and it corresponds to S-wave motion. The second term is the longitudinal one, transmitting with speed $c_{\rm L}$, and it corresponds to P-wave motion. The third term is neither longitudinal nor transversal and it gives contribution arriving at speeds between the two characteristic ones. This shows that this factor represents a combination of P-wave and S-wave motion.

It is also important to note that the following elastodynamic Doppler factors appear in Eq. (25):

$$1 - n_i(t_{\rm ret})V_i(t_{\rm ret})/c \quad \text{for } |\boldsymbol{V}| < c_{\rm T}\,, \qquad c = c_{\rm T}, c_{\rm L}, 1/\kappa \quad \text{and } t_{\rm ret} = t_{\rm T}, t_{\rm L}, t_\kappa\,, \qquad (26)$$

where $n_i = R_i/R$.

4 Radiation of the elastic fields of a point force

In order to derive the elastic fields, we have to calculate the gradient and the time-derivative of the displacement field (25). The appearance of the retarded times (19), (21) and (23) in Eq. (25) makes the differentiation more difficult. To evaluate these derivatives we need some more basic derivatives (see also [3]). We introduce

$$P_c(t') = R(t') - V_m(t')R_m(t')/c\,, \qquad c = c_{\rm T}, c_{\rm L}, 1/\kappa \qquad (27)$$

and obtain after a straightforward calculation

$$\left[\frac{\partial t'}{\partial x_k}\right]\Big|_{t'=t_{\rm ret}} = -\left[\frac{R_k(t')}{cP_c(t')}\right]\Big|_{t'=t_{\rm ret}} \qquad (28)$$

$$\partial_k\big[Q_j(t')\big]\big|_{t'=t_{\rm ret}} = \left[\frac{\partial t'}{\partial x_k}\,\frac{\partial Q_j(t')}{\partial t'}\right]\Big|_{t'=t_{\rm ret}} = -\left[\frac{R_k(t')}{cP_c(t')}\,\dot{Q}_j(t')\right]\Big|_{t'=t_{\rm ret}} \qquad (29)$$

$$\partial_k\left[\frac{1}{P_c(t')}\right]\bigg|_{t'=t_{\rm ret}} = -\frac{1}{P_c^3(t')}\left[\frac{\dot V_m(t')R_m(t')}{c^2}\,R_k(t') + \left(1-\frac{V^2(t')}{c^2}\right)R_k(t') - \frac{P_c(t')}{c}\,V_k(t')\right]\bigg|_{t'=t_{\rm ret}} \tag{30}$$

and

$$\partial_k\left[\frac{R_i(t')R_j(t')}{R^2(t')}\right]\bigg|_{t'=t_{\rm ret}} = \left[\frac{R_j(t')}{R^2(t')}\left(\delta_{ik}+\frac{V_i(t')R_k(t')}{cP_c(t')}\right) + \frac{R_i(t')}{R^2(t')}\left(\delta_{jk}+\frac{V_j(t')R_k(t')}{cP_c(t')}\right) - \frac{2R_i(t')R_j(t')R_k(t')}{R^3(t')P_c(t')}\right]\bigg|_{t'=t_{\rm ret}}, \tag{31}$$

where $\boldsymbol{R}(t')$, $R(t')$, $\boldsymbol{V}(t')$, $P_c(t')$ and $\boldsymbol{Q}(t')$ are to be evaluated at the corresponding retarded times. Using Eqs. (29)–(31), we find for the gradient of the displacement field (25)

$$\begin{aligned}
\beta_{ik}(\boldsymbol{r},t) = -\frac{1}{4\pi\rho}\Bigg\{ & \frac{1}{c_{\rm T}^3}\left[\left(\delta_{ij}-\frac{R_i(t')R_j(t')}{R^2(t')}\right)\frac{R_k(t')\dot Q_j(t')}{P_{\rm T}^2(t')}\right]\bigg|_{t'=t_{\rm T}} + \frac{1}{c_{\rm L}^3}\left[\frac{R_i(t')R_j(t')R_k(t')}{R^2(t')}\,\frac{\dot Q_j(t')}{P_{\rm L}^2(t')}\right]\bigg|_{t'=t_{\rm L}} \\
& + \int_{1/c_{\rm L}}^{1/c_{\rm T}} {\rm d}\kappa\,\kappa^2\left[\left(\frac{3R_i(t')R_j(t')}{R^2(t')}-\delta_{ij}\right)\frac{R_k(t')\dot Q_j(t')}{P_\kappa^2(t')}\right]\bigg|_{t'=t_\kappa} \\
& + \frac{Q_j(t')}{c_{\rm T}^2}\left[\left(\delta_{ij}-\frac{R_i(t')R_j(t')}{R^2(t')}\right)\frac{1}{P_{\rm T}^3(t')}\left(\frac{\dot V_m(t')R_m(t')}{c_{\rm T}^2}R_k(t') + \left(1-\frac{V^2(t')}{c_{\rm T}^2}\right)R_k(t') - \frac{P_{\rm T}(t')}{c_{\rm T}}\,V_k(t')\right)\right. \\
& \left. + \frac{R_j(t')}{R^2(t')P_{\rm T}(t')}\left(\delta_{ik}+\frac{V_i(t')R_k(t')}{c_{\rm T}P_{\rm T}(t')}\right) + \frac{R_i(t')}{R^2(t')P_{\rm T}(t')}\left(\delta_{jk}+\frac{V_j(t')R_k(t')}{c_{\rm T}P_{\rm T}(t')}\right) - \frac{2R_i(t')R_j(t')R_k(t')}{R^3(t')P_{\rm T}^2(t')}\right]\bigg|_{t'=t_{\rm T}} \\
& + \frac{Q_j(t')}{c_{\rm L}^2}\left[\frac{R_i(t')R_j(t')}{R^2(t')}\,\frac{1}{P_{\rm L}^3(t')}\left(\frac{\dot V_m(t')R_m(t')}{c_{\rm L}^2}\,R_k(t') + \left(1-\frac{V^2(t')}{c_{\rm L}^2}\right)R_k(t') - \frac{P_{\rm L}(t')}{c_{\rm L}}\,V_k(t')\right)\right. \\
& \left. - \frac{R_j(t')}{R^2(t')P_{\rm L}(t')}\left(\delta_{ik}+\frac{V_i(t')R_k(t')}{c_{\rm L}P_{\rm L}(t')}\right) - \frac{R_i(t')}{R^2(t')P_{\rm L}(t')}\left(\delta_{jk}+\frac{V_j(t')R_k(t')}{c_{\rm L}P_{\rm L}(t')}\right) + \frac{2R_i(t')R_j(t')R_k(t')}{R^3(t')P_{\rm L}^2(t')}\right]\bigg|_{t'=t_{\rm L}} \\
& + \int_{1/c_{\rm L}}^{1/c_{\rm T}} {\rm d}\kappa\,\kappa\,Q_j(t')\left[\left(\frac{3R_i(t')R_j(t')}{R^2(t')}-\delta_{ij}\right)\frac{1}{P_\kappa^3(t')}\Big(\kappa^2\dot V_m(t')R_m(t')R_k(t')\right. \\
& \qquad + \big(1-\kappa^2V^2(t')\big)R_k(t') - \kappa\,P_\kappa(t')V_k(t')\Big) \\
& \quad - \frac{3R_j(t')}{R^2(t')P_\kappa(t')}\left(\delta_{ik}+\frac{\kappa V_i(t')R_k(t')}{P_\kappa(t')}\right) - \frac{3R_i(t')}{R^2(t')P_\kappa(t')}\left(\delta_{jk}+\frac{\kappa V_j(t')R_k(t')}{P_\kappa(t')}\right) \\
& \quad \left. + \frac{6R_i(t')R_j(t')R_k(t')}{R^3(t')P_\kappa^2(t')}\right]\bigg|_{t'=t_\kappa}\Bigg\}.
\end{aligned} \tag{32}$$

This is the elastic distortion field produced by a non-uniformly moving point force. For a constant strength $Q_j = \text{constant}$, the δ_{ij}-term in the $t_{\rm T}$-expression agrees with the corresponding one given in [3] for the gradient of the Liénard-Wiechert potential in acoustics. The elastic distortion consists of parts depending on the velocity $\boldsymbol{V}$, parts depending on the acceleration $\dot{\boldsymbol{V}}$ (elastic radiation part), and parts depending on $\dot{\boldsymbol{Q}}$. Note that the dot over $\boldsymbol{V}$ and $\boldsymbol{Q}$ indicates a derivative with respect to the argument, namely the corresponding retarded time.

In order to calculate the time derivative of the displacement (25), we also need the relations

$$\left[\frac{\partial t'}{\partial t}\right]\bigg|_{t'=t_{\rm ret}} = \left[\frac{R(t')}{P_c(t')}\right]\bigg|_{t'=t_{\rm ret}} \tag{33}$$

$$\partial_t\big[Q_j(t')\big]\big|_{t'=t_{\rm ret}}=\left[\frac{\partial t'}{\partial t}\,\frac{\partial Q_j(t')}{\partial t'}\right]\bigg|_{t'=t_{\rm ret}}=\left[\frac{R(t')}{P_c(t')}\,\dot{Q}_j(t')\right]\bigg|_{t'=t_{\rm ret}} \tag{34}$$

$$\partial_t\left[\frac{1}{P_c(t')}\right]\bigg|_{t'=t_{\rm ret}}=\frac{1}{P_c^3(t')}\left[\big(\dot{V}_m(t')R_m(t')-V^2(t')\big)\,\frac{R(t')}{c}+V_m(t')R_m(t')\right]\bigg|_{t'=t_{\rm ret}} \tag{35}$$

and

$$\begin{aligned}\partial_t\left[\frac{R_i(t')R_j(t')}{R^2(t')}\right]\bigg|_{t'=t_{\rm ret}}=-\bigg[&\frac{1}{R(t')P_c(t')}\big(V_i(t')R_j(t')+V_j(t')R_i(t')\big)\\&-\frac{2R_i(t')R_j(t')V_m(t')R_m(t')}{R^3(t')P_c(t')}\bigg]\bigg|_{t'=t_{\rm ret}}.\end{aligned} \tag{36}$$

Using Eqs. (34)–(36), we obtain for the time derivative of the displacement field (25)

$$\begin{aligned}v_i(\boldsymbol{r},t)=\frac{1}{4\pi\rho}\Bigg\{&\frac{1}{c_{\rm T}^2}\left[\left(\delta_{ij}R(t')-\frac{R_i(t')R_j(t')}{R(t')}\right)\frac{\dot{Q}_j(t')}{P_{\rm T}^2(t')}\right]\bigg|_{t'=t_{\rm T}}+\frac{1}{c_{\rm L}^2}\left[\frac{R_i(t')R_j(t')}{R(t')}\,\frac{\dot{Q}_j(t')}{P_{\rm L}^2(t')}\right]\bigg|_{t'=t_{\rm L}}\\&+\int_{1/c_{\rm L}}^{1/c_{\rm T}}{\rm d}\kappa\,\kappa\left[\left(\frac{3R_i(t')R_j(t')}{R(t')}-\delta_{ij}R(t')\right)\frac{\dot{Q}_j(t')}{P_\kappa^2(t')}\right]\bigg|_{t'=t_\kappa}\\+\frac{Q_j(t')}{c_{\rm T}^2}\bigg[&\left(\delta_{ij}-\frac{R_i(t')R_j(t')}{R^2(t')}\right)\frac{1}{P_{\rm T}^3(t')}\left(\big[\dot{V}_m(t')R_m(t')-V^2(t')\big]\frac{R(t')}{c_{\rm T}}+V_m(t')R_m(t')\right)\\&+\frac{1}{R(t')P_{\rm T}^2(t')}\big(V_i(t')R_j(t')+V_j(t')R_i(t')\big)-\frac{2R_i(t')R_j(t')V_m(t')R_m(t')}{R^3(t')P_{\rm T}^2(t')}\bigg]\bigg|_{t'=t_{\rm T}}\\+\frac{Q_j(t')}{c_{\rm L}^2}\bigg[&\frac{R_i(t')R_j(t')}{R^2(t')}\frac{1}{P_{\rm L}^3(t')}\left(\big[\dot{V}_m(t')R_m(t')-V^2(t')\big]\frac{R(t')}{c_{\rm L}}+V_m(t')R_m(t')\right)\\&-\frac{1}{R(t')P_{\rm L}^2(t')}\big(V_i(t')R_j(t')+V_j(t')R_i(t')\big)+\frac{2R_i(t')R_j(t')V_m(t')R_m(t')}{R^3(t')P_{\rm L}^2(t')}\bigg]\bigg|_{t'=t_{\rm L}}\\+\int_{1/c_{\rm L}}^{1/c_{\rm T}}&{\rm d}\kappa\,\kappa\,Q_j(t')\bigg[\left(\frac{3R_i(t')R_j(t')}{R^2(t')}-\delta_{ij}\right)\frac{1}{P_\kappa^3(t')}\Big(\big[\dot{V}_m(t')R_m(t')-V^2(t')\big]\kappa R(t')+V_m(t')R_m(t')\Big)\\&-\frac{3}{R(t')P_\kappa^2(t')}\big(V_i(t')R_j(t')+V_j(t')R_i(t')\big)+\frac{6R_i(t')R_j(t')V_m(t')R_m(t')}{R^3(t')P_\kappa^2(t')}\bigg]\bigg|_{t'=t_\kappa}\Bigg\}.\end{aligned} \tag{37}$$

This is the velocity field (particle velocity) produced by a non-uniformly moving point force. For a constant strength $Q_j=$ constant, the δ_{ij}-term in the $t_{\rm T}$-expression agrees with the corresponding one given in [3] for the time derivative of the Liénard-Wiechert potential in acoustics. Again the velocity vector consists of parts depending on the velocity $\boldsymbol{V}$, fields depending on the acceleration $\dot{\boldsymbol{V}}$ (radiation part), and fields depending on $\dot{\boldsymbol{Q}}$. The dot over $\boldsymbol{V}$ and $\boldsymbol{Q}$ indicates again a derivative with respect to the argument, namely the corresponding retarded time.

5 The Stokes solution as limit of a non-uniformly moving point force

In the current section, we show that the Stokes solution is contained in our general results for a non-uniformly moving point force. In this sense, our solutions (25) and (32) are the correct

generalizations of the Stokes solution towards the non-uniform motion. If the position of the point force is fixed, which means that $\boldsymbol{s}$ is time-independent and therefore $\boldsymbol{V} = 0$, we recover from the displacement (25) the famous Stokes solution of a concentrated point force with time-dependent magnitude (e.g. [11, 9])

$$u_i(\boldsymbol{r},t) = \frac{1}{4\pi\rho R}\left\{\frac{1}{c_{\mathrm{T}}^2}\left(\delta_{ij} - \frac{R_iR_j}{R^2}\right)Q_j(t-R/c_{\mathrm{T}}) + \frac{1}{c_{\mathrm{L}}^2}\,\frac{R_iR_j}{R^2}\,Q_j(t-R/c_{\mathrm{L}}) \right.$$
$$\left. + \left(\frac{3R_iR_j}{R^2} - \delta_{ij}\right)\int_{1/c_{\mathrm{L}}}^{1/c_{\mathrm{T}}} \kappa\, Q_j(t-\kappa R)\,\mathrm{d}\kappa\right\}. \tag{38}$$

The first terms in Eq. (38) are usually called the far-field terms since they behave as $1/R$ and the last term in Eq. (38) is called the near-field term (see, e.g, [2, 20]). From Eq. (32) and after some mathematical manipulations, we find the corresponding displacement gradient of the Stokes solution (e.g. [11, 9])

$$\beta_{ik}(\boldsymbol{r},t) = -\frac{1}{4\pi\rho}\left\{3\left(\frac{5R_iR_jR_k}{R^5} - \frac{\delta_{ij}R_k + \delta_{jk}R_i + \delta_{ik}R_j}{R^3}\right)\int_{1/c_{\mathrm{L}}}^{1/c_{\mathrm{T}}} \kappa\, Q_j(t-\kappa R)\,\mathrm{d}\kappa \right.$$
$$+ \left(\frac{6R_iR_jR_k}{R^5} - \frac{\delta_{ij}R_k + \delta_{jk}R_i + \delta_{ik}R_j}{R^3}\right)\left[\frac{1}{c_{\mathrm{L}}^2}\,Q_j(t-R/c_{\mathrm{L}}) - \frac{1}{c_{\mathrm{T}}^2}\,Q_j(t-R/c_{\mathrm{T}})\right]$$
$$+ \frac{\delta_{ij}R_k}{c_{\mathrm{T}}^2R^3}\left[Q_j(t-R/c_{\mathrm{T}}) + \frac{R}{c_{\mathrm{T}}}\,\dot{Q}_j(t-R/c_{\mathrm{T}})\right]$$
$$\left. + \frac{R_iR_jR_k}{R^4}\left[\frac{1}{c_{\mathrm{L}}^3}\,\dot{Q}_j(t-R/c_{\mathrm{L}}) - \frac{1}{c_{\mathrm{T}}^3}\,\dot{Q}_j(t-R/c_{\mathrm{T}})\right]\right\}. \tag{39}$$

Thus, Eqs. (25) and (32) give the correct Stokes solution as limit. It can be seen that the $\dot{\boldsymbol{Q}}$-terms in Eq. (39) behave as $1/R$ (far-field terms or radiation terms) and the $\boldsymbol{Q}$-terms behave as $1/R^2$ (near-field terms).

6 Static limit of a non-uniformly moving point force

In this section, we give the static limit of the Liénard-Wiechert potential and the elastic distortion of a point force as a further check of the obtained results.

For the static limit, we set $\boldsymbol{V} = 0$ and $\boldsymbol{Q} =$ constant and substitute Eq. (12) and $\lambda = 2\mu\nu/(1-2\nu)$ in Eq. (25). If we perform the integration in κ and arrange in proper order the appearing terms, we recover the displacement field of the Kelvin problem (see, e.g., [11])

$$u_i(\boldsymbol{r}) = \frac{Q_j}{16\pi\mu(1-\nu)R}\left[(3-4\nu)\delta_{ij} + \frac{R_iR_j}{R^2}\right], \tag{40}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and ν is Poisson's ratio. From Eq. (32), we find in the static limit the displacement gradient of the Kelvin problem (see, e.g., [11])

$$\beta_{ik}(\boldsymbol{r}) = -\frac{Q_j}{16\pi\mu(1-\nu)R^3}\left[(3-4\nu)\delta_{ij}R_k - \delta_{ik}R_j - \delta_{jk}R_i + \frac{3R_iR_jR_k}{R^2}\right]. \tag{41}$$

Thus, in the static limit, we recovered the displacement and displacement gradient fields of the Kelvin problem, which is concerned with a concentrated point force in three-dimensional elastostatics.

7 The elastodynamic Liénard-Wiechert potentials and elastic fields of non-uniformly moving line forces

We proceed to derive the two-dimensional Liénard-Wiechert potentials and the elastic fields of non-uniformly moving line forces. We consider line forces moving non-uniformly and with time-dependent magnitude. The line forces are parallel to the x_3-direction. In two dimensions, the in-plane line force is given by

$$F_\alpha = Q_\alpha(t)\,\delta(\boldsymbol{R}(t))\,, \tag{42}$$

and the anti-plane line force reads

$$F_3 = Q_3(t)\,\delta(\boldsymbol{R}(t))\,, \tag{43}$$

where $\boldsymbol{R}(t) = \boldsymbol{r} - \boldsymbol{s}(t) \in \mathbb{R}^2$ and $\alpha = 1, 2$. The field variables are independent of the x_3-component.

If the material is infinitely extended and isotropic, the two-dimensional elastodynamic Green tensor of plane-strain reads [9, 14]

$$\begin{aligned} G_{\alpha\beta}(\boldsymbol{r}, t) = \frac{1}{2\pi\rho}\bigg\{ & \frac{x_\alpha x_\beta}{r^4}\bigg(\frac{\big[2t^2 - r^2/c_{\mathrm{L}}^2\big]}{\sqrt{t^2 - r^2/c_{\mathrm{L}}^2}}\, H\big(t - r/c_{\mathrm{L}}\big) - \frac{\big[2t^2 - r^2/c_{\mathrm{T}}^2\big]}{\sqrt{t^2 - r^2/c_{\mathrm{T}}^2}}\, H\big(t - r/c_{\mathrm{T}}\big)\bigg) \\ & - \frac{\delta_{\alpha\beta}}{r^2}\bigg(\sqrt{t^2 - r^2/c_{\mathrm{L}}^2}\, H\big(t - r/c_{\mathrm{L}}\big) - \frac{t^2}{\sqrt{t^2 - r^2/c_{\mathrm{T}}^2}}\, H\big(t - r/c_{\mathrm{T}}\big)\bigg)\bigg\} \end{aligned} \tag{44}$$

and the elastodynamic Green tensor of anti-plane strain is given by

$$G_{33}(\boldsymbol{r}, t) = \frac{1}{2\pi\rho c_{\mathrm{T}}^2}\, \frac{H\big(t - r/c_{\mathrm{T}}\big)}{\sqrt{t^2 - r^2/c_{\mathrm{T}}^2}}\,, \tag{45}$$

where $H(.)$ denotes the Heaviside step function and $r = \sqrt{x_1^2 + x_2^2}$.

If we substitute Eqs. (44) and (42) in Eq. (14) and perform the integration in $\boldsymbol{r}'$, we find for the displacement field

$$\begin{aligned} u_\alpha(\boldsymbol{r}, t) = \frac{1}{2\pi\rho}\bigg[& \int_{-\infty}^{t_{\mathrm{L}}} Q_\beta(t')\bigg(\frac{R_\alpha(t')R_\beta(t')}{R^4(t')}\, \frac{\bar{t}^2}{S_{\mathrm{L}}(t')} + \bigg(\frac{R_\alpha(t')R_\beta(t') - \delta_{\alpha\beta}\, R^2(t')}{R^4(t')}\bigg) S_{\mathrm{L}}(t')\bigg)\mathrm{d}t' \\ & - \int_{-\infty}^{t_{\mathrm{T}}} Q_\beta(t')\bigg(\frac{R_\alpha(t')R_\beta(t')}{R^4(t')}\, S_{\mathrm{T}}(t') + \bigg(\frac{R_\alpha(t')R_\beta(t') - \delta_{\alpha\beta}\, R^2(t')}{R^4(t')}\bigg) \frac{\bar{t}^2}{S_{\mathrm{T}}(t')}\bigg)\mathrm{d}t'\bigg]\,. \end{aligned} \tag{46}$$

The notation here is

$$\bar{t} = t - t'\,, \qquad S_{\mathrm{T}}^2(t') = \bar{t}^2 - \frac{R^2(t')}{c_{\mathrm{T}}^2}\,, \qquad S_{\mathrm{L}}^2(t') = \bar{t}^2 - \frac{R^2(t')}{c_{\mathrm{L}}^2}\,. \tag{47}$$

The two retarded times t_{T} and t_{L} are the roots of $S_{\mathrm{T}}^2(t') = 0$ and $S_{\mathrm{L}}^2(t') = 0$, respectively, which are less than t. The solving of the conditions $S_{\mathrm{T}}^2(t') = 0$ and $S_{\mathrm{L}}^2(t') = 0$ is non-trivial for a general motion and can be complicated. For a subsonic motion, the solutions for the retarded

times $t_{\rm T}$ and $t_{\rm L}$ are unique. The corresponding elastic distortion and velocity of a non-uniformly moving line force are

$$\begin{aligned}\beta_{\alpha\gamma}(\boldsymbol{r},t)=\frac{1}{2\pi\rho}\,\partial_\gamma\bigg[\int_{-\infty}^{t_{\rm L}}Q_\beta(t')\bigg(\frac{R_\alpha(t')R_\beta(t')}{R^4(t')}\,\frac{\bar{t}^2}{S_{\rm L}(t')}+\bigg(\frac{R_\alpha(t')R_\beta(t')-\delta_{\alpha\beta}\,R^2(t')}{R^4(t')}\bigg)S_{\rm L}(t')\bigg){\rm d}t'\\
-\int_{-\infty}^{t_{\rm T}}Q_\beta(t')\bigg(\frac{R_\alpha(t')R_\beta(t')}{R^4(t')}\,S_{\rm T}(t')+\bigg(\frac{R_\alpha(t')R_\beta(t')-\delta_{\alpha\beta}\,R^2(t')}{R^4(t')}\bigg)\frac{\bar{t}^2}{S_{\rm T}(t')}\bigg){\rm d}t'\bigg]\end{aligned}\tag{48}$$

and

$$\begin{aligned}v_{\alpha}(\boldsymbol{r},t)=\frac{1}{2\pi\rho}\,\partial_t\bigg[\int_{-\infty}^{t_{\rm L}}Q_\beta(t')\bigg(\frac{R_\alpha(t')R_\beta(t')}{R^4(t')}\,\frac{\bar{t}^2}{S_{\rm L}(t')}+\bigg(\frac{R_\alpha(t')R_\beta(t')-\delta_{\alpha\beta}\,R^2(t')}{R^4(t')}\bigg)S_{\rm L}(t')\bigg){\rm d}t'\\
-\int_{-\infty}^{t_{\rm T}}Q_\beta(t')\bigg(\frac{R_\alpha(t')R_\beta(t')}{R^4(t')}\,S_{\rm T}(t')+\bigg(\frac{R_\alpha(t')R_\beta(t')-\delta_{\alpha\beta}\,R^2(t')}{R^4(t')}\bigg)\frac{\bar{t}^2}{S_{\rm T}(t')}\bigg){\rm d}t'\bigg]\,.\end{aligned}\tag{49}$$

Now we consider the anti-plane line force. If we substitute Eqs. (45) and (43) in Eq. (14) and perform the integration in $\boldsymbol{r}'$, we find for the displacement field of a line load of body forces pointing in the x_3-direction

$$u_3(\boldsymbol{r},t)=\frac{1}{2\pi\rho c_{\rm T}^2}\int_{-\infty}^{t_{\rm T}}\frac{Q_3(t')}{S_{\rm T}(t')}\,{\rm d}t'\,.\tag{50}$$

The corresponding elastic distortion and velocity fields are

$$\beta_{3\alpha}(\boldsymbol{r},t)=\frac{1}{2\pi\rho c_{\rm T}^2}\,\partial_\alpha\int_{-\infty}^{t_{\rm T}}\frac{Q_3(t')}{S_{\rm T}(t')}\,{\rm d}t'\,,\qquad v_3(\boldsymbol{r},t)=\frac{1}{2\pi\rho c_{\rm T}^2}\,\partial_t\int_{-\infty}^{t_{\rm T}}\frac{Q_3(t')}{S_{\rm T}(t')}\,{\rm d}t'\,.\tag{51}$$

The displacement fields of non-uniformly moving line forces play the role of the two-dimensional elastodynamical Liénard-Wiechert potentials. The two-dimensional Liénard-Wiechert potentials (46) and (50) are time-integrals over the history of the motion and, thus, they are characterized by an afterglow. Such two-dimensional wave motion possesses a 'tail' characteristic for the so-called diffusion of waves. For that reason line forces are haunted by their past. Nevertheless, for the two-dimensional Liénard-Wiechert potentials (46) and (50) we are left to evaluate time-integrals of considerable complexity, which only in some simple cases yield results of elementary functions in a closed form. Also, the calculation of the retarded times is not a trivial task. The mathematical complexity of the integrals (46)–(51) is the same as of the integral expressions of non-uniformly moving straight dislocations given by Lardner [16] and Lazar [17].

If we put $s(t')=0$ in Eqs. (46) and (50), we recover the displacement fields of concentrated line forces given by de Hoop [6] and Achenbach [1]. For the in-plane line force, de Hoop [6] and Achenbach [1] used a more complicated but equivalent representation of the Green tensor (44).

8 Conclusion

Exact analytical solutions of the displacement and of elastic fields have been calculated for point and line forces moving non-uniformly in an unbounded, elastic, isotropic body. We have investigated the subsonic motion ($|\boldsymbol{V}|<c_{\rm T}$). We have shown that the displacements can be interpreted as the elastodynamic Liénard-Wiechert potentials caused by body forces. For a

point force, we calculated explicitly the radiation parts of the elastic fields. We have proven that our solution of a non-uniformly moving point force is the correct generalization of the Stokes solution and of the solution of the Kelvin problem. In the case of line forces the Liénard-Wiechert potentials are given in the form of time-integral representations, which cannot be further simplified for the general non-uniform motion.

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References

[1] J.D. Achenbach, *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam (1973).

[2] K. Aki and P.G. Richards, *Quantitative Seismology*, 2nd Edition, University Science Books, Sausalito, California (2002).

[3] G. Barton, *Elements of Green's Functions and Propagation*, Oxford University Press, Oxford (1989).

[4] A. Ben-Menahem and S. Singh, *Seismic Waves and Sources*, Springer Verlag, New York (1981).

[5] C. Chapman, *Fundamentals of Seismic Wave Propagation*, Cambridge University Press, Cambridge (2004).

[6] A.T. de Hoop, *Representation theorems for the displacement in an elastic solid and their applications to elastodynamic diffraction theory*, Doctoral Dissertation, Delft (1958).

[7] A.T. de Hoop, *Handbook of Radiation and Scattering of Waves*, Academic Press, London (1995).

[8] A.T. de Hoop, Fields and waves excited by impulsive point sources in motion – The general 3D time-domain Doppler effect, Wave Motion **42** (2005) 116–122.

[9] A.C. Eringen and S.S. Suhubi, *Elastodynamics, Volume II, Linear Theory*. Academic Press, New York (1975).

[10] L.B. Freund, Wave motion in an elastic solid due to a non-uniform moving line load, Q. Appl. Math. **30** (1972) 271–281.

[11] M.E. Gurtin, The linear theory of elasticity, in: *Handbuch der Physik VIa/2*, S. Flügge, ed., Springer, Berlin (1972) pp. 1–345.

[12] J.A. Hudson, *The Excitation and Propagation of Elastic Waves*, Cambridge University Press, Cambridge (1980).

[13] J.D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley, New York (1999).

[14] E. Kausel, *Fundamental Solutions in Elastodynamics*, Cambridge University Press, Cambridge (2006).

[15] L.D. Landau and E.M. Lifschitz, *The Classical Theory of Fields*, 4th ed., Pergamon, Oxford (1987).

[16] R.W. Lardner, *Mathematical Theory of Dislocations and Fracture*, University of Toronto Press, Toronto (1974).

[17] M. Lazar, On the elastic fields produced by non-uniformly moving dislocations: a revisit, Philosophical Magazine **91** (2011) 3327–3342.

[18] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York (1944).

[19] J. Miklowitz, *Elastic Waves and Waveguides*, North-Holland, Amsterdam (1978).

[20] J. Pujol, *Elastic Wave Propagation and Generation in Seismology*, Cambridge University Press, Cambridge (2003).

[21] B.W. Roos, *Analytical Functions and Distributions in Physics and Engineering*, Wiley, New York (1969).

[22] G.G. Stokes, On the dynamical theory of diffraction, Trans. Cambridge Philos. Soc. **9** (1849) 1–66.

[23] K.-C. Wu, A non-uniformly moving line force in an anisotropic elastic solid, Proc. R. Soc. Lond. A **458** (2002) 1761–1772.